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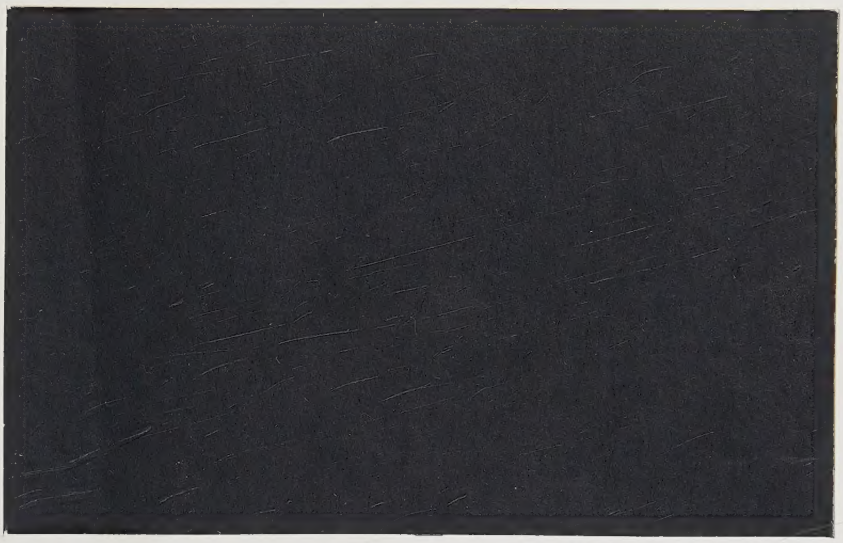

DEPARTMENT OF MATHEMATICS

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POWER BOUNDED RESTRICTIONS OF FOURIER-STIELTJES TRANSFORMS

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för filosofie doktorsexamen i matematik (examinator Philip Brenner), som enligt beslut av matematisk/teknisk fysiska sektionsgruppsnämnden vid Göteborgs universitet och Chalmers tekniska högskola kommer att offentligt försvaras lördagen den 11 mars kl. 10.15 i sal F vid matematiska institutionen, Sven Hultins gata 6, Göteborg.

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1. Introduction.

In the present paper we give local variants of the theorems of Beurling-Helson and Cohen, i.e. characterizations of the power bounded elements in certain restriction algebras of Fourier-Stieltjes transforms on locally compact abelian groups. It is well known that there are no obvious such characterizations in general. However, we will provide a characterization, presented in some detail below, in the case when the algebra consists of restrictions to a set E which is compact and the closure of its interior. We also obtain some results in the case when E is the closure of an open connected set.

Let G be a locally compact abelian group with dual group Γ and $M(\Gamma)$ be the convolution algebra of all complex valued regular measures μ on Γ such that $\|\mu\|_{M(\Gamma)}$, the total variation of μ is finite. The Fourier-Stieltjes transform of $\mu \in M(\Gamma)$ is defined by

$$\hat{\mu}(x) = \int_{\Gamma} \overline{\langle x,\gamma \rangle} d\mu(\gamma).$$

For a closed set E in G we denote by $B(E)$ the algebra of restrictions of Fourier-Stieltjes transforms to E. With norm defined by

$$\|f\|_{B(E)} = \inf\{\|\mu\|_{M(\Gamma)} \mid \hat{\mu} = f \text{ on } E\},$$

$B(E)$ is a Banach algebra of continuous functions. The set of all functions $f \in B(E)$ with $|f| = 1$ on E and $\sup_{n \in \mathbb{Z}_+} \|f^n\|_{B(E)} < \infty$ will be denoted by $\mathcal{B}(E)$. By the well known theorem of Beurling and Helson [1], for G connected, any function in $\mathcal{B}(G)$ is a character times a complex constant of modulus one. A more general result due to Cohen, (See [4], Theorem 1, [12], Theorem 4.7.3) gives a characterization of the functions in $\mathcal{B}(G)$, valid for any G. The main result of this paper is a similar characterization of the functions in $\mathcal{B}(E)$ when E is compact and the closure of its interior. We will now describe this result.

Let T be the group of complex numbers of modulus one and U an open neighbourhood of the identity in G. A continuous map $\theta: U \to T$ is called a local character if whenever $x \in U$, $y \in U$ and $x + y \in U$ then $\theta(x+y) = \theta(x)\,\theta(y)$.

If $\overline{U}$ is compact then the continuous extrapolation to $\overline{U}$ of θ is in $\mathcal{B}(\overline{U})$ (cf section 2). We denote by $\mathcal{L}(E)$ the set of functions that locally on E agrees with translates of local characters multiplied by complex numbers of modulus one. When E is compact, it follows by a partition of the unit that $\mathcal{L}(E) \subset \mathcal{B}(E)$. Our characterization is that if E is compact and the closure of its interior then $\mathcal{B}(E) = \mathcal{L}(E)$. For such sets E we also give a more explicit description of the functions in $\mathcal{L}(E)$. (cf section 5).

We start by proving that if V is any open set in G and $f \in \mathcal{B}(\overline{V})$ then $f|_V \in \mathcal{L}(V)$. This is done by structure theory using the above-mentioned result of Cohen and a local version of the theorem of Beurling and Helson [3] in $\mathbb{R}^n$. Using a technique introduced by Domar in [4] we then improve this result, showing that for some integer p not exceeding $\sup_{n \in \mathbb{Z}_+} \|f^n\|_{B(\overline{V})}$, f agrees locally on V with translates of just p essentially different local characters multiplied by complex numbers of modulus one. This gives a natural partition of V in p open subsets V_k, f being locally equal to a translate of one and the same local character times a complex number of modulus one on V_k. Then, if x is a boundary point of some V_k we choose a small neighborhood W of the identity in G with $\overline{W}$ compact and divide f by the x-translate of the local character. For sufficiently small W, the quotient is in $\mathcal{B}(\overline{V_k \cap (x+W)})$ and is locally constant in $V_k \cap (x+W)$. The proof is finished by showing that such a function has a finite range.

In section 6 we first give a few examples which show that there is no general characterization of the functions in $\mathcal{B}(E)$, when E is the closure of an open set. However, if E is the closure of an open and connected set, the functions in $\mathcal{B}(E)$ admit a simple description, similar to that given by the theorem of Beurling and Helson, (cf. section 6). The section is finished with a discussion of for what sets E, functions of this form are in $\mathcal{B}(E)$.

In submitting this thesis, I wish to thank Docent Philip Brenner for suggesting the topic and for his kind interest in my work. I also wish to express my gratitude to Professor Yngve Domar, whose work in the area of harmonic analysis has

been a great source of inspiration. Some work on the problems discussed here benefited from time spent at Institute Mittag-Leffler the academic year 1973-1974 and in the spring 1977.

2. Preliminaries.

We begin by reviewing certain notions concerning topological groups. Any group considered in this paper, unless the contrary is explicitly stated, will be locally compact and abelian (abbreviated LCA). Except for T, the group of complex numbers of modulus one, all groups will be written additively with 0 as identity element.

A homomorphism is said to be a morphism if continuous and an epimorphism if it is continuous, open and surjective. If U is an open neighbourhood (shortened open nbd) of 0 in G, a continuous map θ of U into a group H is called a local morphism if whenever $x \in U$, $y \in U$ and $x + y \in U$ then $\theta(x+y) = \theta(x) + \theta(y)$. A local isomorphism of G with H is a homeomorphism θ of a nbd of 0 in G onto a nbd of 0 in H such that θ and θ^{-1} are local morphisms. When such a map exists the groups are said to be locally isomorphic. Morphisms (local morphisms) into T are of course named characters (local characters).

In general a local character is not a restriction of a character. However the following device (cf eg. [10], Theorem 18) enable us regard it as induced by a character of a "larger" group.

Theorem 2.1. Let $\theta: U \to T$ be a local character in G, G_0 be the open subgroup of G generated by U and $\widetilde{U} = \{(x, \theta(x)) \in G \times T \mid x \in U\}$. Let $\widetilde{G}$ be the subgroup of $G \times T$ generated by $\widetilde{U}$ and (disregarding its topology as a subgroup of $G \times T$) topologized by taking as a base for nbds of the origin the sets $\widetilde{V} = \{(x,\theta(x)) \in G \times T \mid x \in V \subset U\}$, V ranging over the open nbds of 0 in G. If α and β are the restrictions to $\widetilde{G}$ of the projection of $G \times T$ onto G and of the projection of $G \times T$ onto T, respectively, then

(1) $\widetilde{G}$ is LCA

(2) $\alpha: \widetilde{G} \to G_0$ is an epimorphism and $\alpha|_{\widetilde{U}}: \widetilde{U} \to U$ is a homeomorphism, whence $\widetilde{G}$ and G are locally isomorphic

(3) β is a character of $\widetilde{G}$ and $\theta = \beta \circ (\alpha|_{\widetilde{U}})^{-1}$. ∎

Remark 1. The group $\widetilde{G}$ is minimal in the following sence: if G', U', α', β'

satisfies (1), (2) and (3) above then $\widetilde{G}$ is isomorphic with an open subgroup G' modulo a discrete subgroup.

Remark 2. Let $\alpha: \mathbb{R}^q \to \mathbb{R}^q/\mathbb{Z}^q = \mathbb{T}^q$ be the canonical map, U an open connected nbd of 0 in $\mathbb{R}^q$ contained in $]-\frac{1}{4}, \frac{1}{4}[^q$ and $V = \alpha(U)$. For any $a \in \mathbb{R}^q$ $\theta_a: x \to e^{2\pi i \langle a, (\alpha|_U)^{-1}(x)\rangle}$, $x \in V$, is a local character on V ($\langle , \rangle$ is the usual inner product in $\mathbb{R}^q$). Conversely any local character on V is of this form, since a local morphism in $\mathbb{R}^q$ coincides on some nbd of 0 with a morphism ([2], p. 88). It is readily seen that in this case $\widetilde{\mathbb{R}}^q$ is (isomorphic with) $\mathbb{R}^q/\mathbb{R}^q \cap (a)^\perp$, where $(a)^\perp$ denotes the annihilator of the subgroup generated by a.

In 3 we will need a certain type of basis at 0 in compact abelian groups. Much more general results of this type are known ([10], Theorem 69), but for the convenience of the reader we include a direct proof of the special case we will use.

Theorem 2.2. For each nbd U of 0 in a compact abelian group G there is a closed subgroup H, a morphism $\psi: \mathbb{R}^q \to G$ for some $q \geq 0$ (with $\mathbb{R}^0 = \{0\}$) and an open connected nbd V of 0 in $\mathbb{R}^q$ such that

(1) $\psi(V) + H$ is open and contained in U

(2) the map $(t,h) \to \psi(t) + h$, $(t,h) \in V \times H$ is a local isomorphism.

Proof. Since the dual group Γ of G is discrete there are $\gamma_1, \dots, \gamma_p$ in Γ and $\varepsilon > 0$ such that U contains the open set

$$[illegible]$$

The subgroup Λ of Γ generated by $\gamma_1, \dots, \gamma_p$ is isomorphic with a quotient group of $\mathbb{Z}^p$ and can thus be written as a direct sum, $\Lambda = \Lambda_1 \oplus \Lambda_2$ where Λ_1 is isomorphic with $\mathbb{Z}^q$ and Λ_2 is a finite group. We may assume that ε is so small that $N(p,\varepsilon) \subset \Lambda_2^\perp$, the annihilator of Λ_2. Let $H = \Lambda^\perp$. If $q = 0$ then $H = N(p,\varepsilon)$ and (1) and (2) are trivially satisfied. If $q > 0$ we may also assume that $\gamma_1, \dots, \gamma_q \in \Lambda_1$ that $\gamma_{q+1}, \dots \gamma_p \in \Lambda_2$ and that for a suitable isomorphism $\sigma: \Lambda_1 \to \mathbb{Z}^q$, $\sigma(\gamma_j) = e_j$, where e_j is the element in $\mathbb{Z}^q$ whose j:th coordinate

is 1 and the others are 0. Let $\tau: \mathbb{Z}^q \to \mathbb{R}^q$ be the inclusion map. As $\mathbb{R}^q$ is divisible and Γ is discrete, we can extend $\tau \circ \sigma$ to a morphism ρ of Γ into $\mathbb{R}^q$. Let $\psi: \mathbb{R}^q \to G$ be the dual of ρ, i.e.

$$\forall \gamma \in \Gamma \quad \forall t \in \mathbb{R}^q \quad \langle \psi(t), \gamma \rangle = e^{2\pi i \langle \rho(\gamma), t \rangle},$$

and let

$$V = \{t = (t_1, \ldots, t_q) \in \mathbb{R}^q \mid |t_j| < \frac{1}{\pi} \arcsin \frac{\varepsilon}{2},\ 1 \leq j \leq q\}.$$

Now if ε is small and $x \in N(p,\varepsilon)$, there is a unique $t = (t_1, \ldots, t_q) \in V$ such that $\langle x, \gamma_j \rangle = e^{2\pi i t_j}$ for $1 \leq j \leq q$ and thus, $x - \psi(t) \in \Lambda_1^{\perp}$. But $x - \psi(t) \in \Lambda_2^{\perp}$ too, since $N(p,\varepsilon) \subset \Lambda_2^{\perp}$ and $\rho(\Lambda_2) = \{0\}$. Hence $N(p,\varepsilon) = \psi(V) + H$ and the map $(t,h) \to \psi(t) + h$, $(t,h) \in V \times H$ is a homeomorphism, as the argument above also shows that its continuous extension to the compact $\overline{V} \times H$ is injective. ∎

Remark. If $q = 0$ for each nbd U of 0 in G, then G is totally disconnected and conversely.

For Fourier analysis we will in general follow [12] but with the roles of G and Γ reversed, thus taking Fourier and Fourier-Stieltjes transforms on G of functions and measures on Γ. We will frequently use the following immediate consequence of Lemma 4.2.2 in [12].

Lemma 2.1. If α is an affine map of the group G_1 into the group G_2, if $E \subset G_2$ is closed and $f \in B(E)$ then $f \circ \alpha \in B(\alpha^{-1}(E))$ and $\|f\circ\alpha\|_{B(\alpha^{-1}(E))} \leq \|f\|_{B(E)}$. ∎

The next result, due to de Leeuw and Herz [5] enable us to go in the opposite direction.

Theorem 2.3. If α is a morphism of the group G_1 into the group G_2, if $E \subset G_1$ is compact and α is injective on a nbd of E, then there is a constant C such that if $f \in B(E)$ then $f \circ (\alpha|_E)^{-1} \in B(\alpha(E))$ and $\|f\circ(\alpha|_E)^{-1}\|_{B(\alpha(E))} \leq C\|f\|_{B(E)}$.

In particular this theorem and Theorem 2.1. imply that if U is an open nbd of 0 with $\overline{U}$ compact, the continuous extrapolation to U of a local character on U is in $\mathcal{B}(\overline{U})$.

For a closed set E in G we denote by $A(E)$ the algebra of restrictions to E of Fourier transforms of functions in $L^1(\Gamma)$. Thus $f \in A(E)$ if there is

a $\phi \in L^1(\Gamma)$ such that $f(x) = \hat{\phi}(x) = \int_\Gamma \overline{<x,\gamma>}\ \phi(\gamma)d\gamma$, $x \in E$ and $\|f\|_{A(E)} = \inf\{\|\phi\|_{L^1(\Gamma)} \mid \hat{\phi} = f \text{ on } E\}$. We finally remind that if E is compact and $\varepsilon > 0$ there is a $k \in L^1(\Gamma)$ with $\hat{k} = 1$ on E. Thus for compact sets E, we have $B(E) = A(E)$ and with the same norms.

3. A local result.

Let E be a closed set in a LCA group G and $\mathcal{B}(E)$ the set of all functions $f \in B(E)$ with $|f| = 1$ on E and $\sup_{n \in \mathbb{Z}_+} \|f^n\|_{B(E)} < \infty$. Though our main results on $\mathcal{B}(E)$ require E to be compact or connected, in this and the next section we only assume E to be a closed set with a non-empty interior.

Theorem 3.1. If $f \in \mathcal{B}(E)$ then each interior point in E has a nbd in which f equals a translate of some local character times a constant of modulus one.

The essential step in the proof is the following lemma.

Lemma 3.1. Let W be an open connected nbd of 0 in $\mathbb{R}^p$ $p > 0$ and H a compact group. If $g \in \mathcal{B}(\overline{W} \times H)$ then there is a piecewise affine map $C: H \to T$ and a piecewise constant map $a: H \to \mathbb{R}^p$ such that

$$g(t,h) = C(h)e^{2\pi i <a(h),t>} \qquad (t,h) \in \overline{W} \times H.$$

We follow [12] in the definitions and properties of affine and piecewise affine maps. For the convenience of the reader, we repeat these definitions here.

Let G_1 and G_2 be LCA groups and E a coset in G_2, i.e. a translate of a subgroup of G_2. If $\alpha: E \to G_1$ is continuous map such that

$$\alpha(x+x'-x'') = \alpha(x) + \alpha(x') - \alpha(x''),\ x, x', x'' \in E$$

then α is said to be affine. If S_i, $1 \le i \le m$, are pairwise disjoint sets belonging to the coset ring of G_2, each S_i contained in an open coset K_i of G_2 and $\alpha_i: K_i \to G_1$ is affine for each i, then the map α of $\bigcup_{i=1}^m S_i$ into G_1 which coincides on S_i with α_i is said to be piecewise affine. We remind that the coset ring of G_2 is the smallest ring of subsets of G_2 containing all open cosets in G_2. Piecewise constant maps are defined similarly with obvious modifications.

Proof of the lemma. First fix $h \in H$ and consider the function $g_h: t \to g(t,h)$ on $\bar{W}$. As the map $t \to (t,h)$, $t \in \mathbb{R}^p$ is affine, $g_h \in \mathcal{B}(\bar{W})$ by Lemma 2.1. By Theorem 1 in [3], which is correct for $\Gamma = \mathbb{R}^n$, there are $C(h) \in T$ and $a(h) \in \mathbb{R}^p$ such that $g_t(h) = C(h)\, e^{2\pi i <a(h),t>}$, $t \in \bar{W}$, thus defining the maps C and a. On the other hand for $t \in \bar{W}$ fixed, the function $g_t: h \to g(t,h)$ is in $\mathcal{B}(H)$ again by Lemma 2.1. By the result of Cohen (cf. [4], Theorem 1, [12], Theorem 4.7.3) the function g_t is piecewise affine. Choosing $t = 0$ we see that C is piecewise affine, hence so are all the functions $h \to e^{2\pi i <a(h),t>}$. This implies a to be piecewise affine. But a piecewise affine map from a compact group into $\mathbb{R}^p$ is piecewise constant, since the only compact subgroup of $\mathbb{R}^p$ if $\{0\}$.

Remark. If a non-empty set S belongs to the coset ring of H, then it belongs to the smallest ring of subsets of H containing all cosets of some finite family of open subgroups of H. Hence, as H is compact, S is simply a union of cosets of an open subgroup of H. Thus the conclusion of Lemma 3.1. could be expressed more explicitly as follows: there are disjoint sets $S_1, \ldots, S_m$, each a union of cosets of some fixed open subgroup of H, with union H and $c_j \in T$, $a_j \in \mathbb{R}^p$ and $\gamma_j \in \hat{H}$ such that

$$g(t,h) = c_j\, \gamma_j(h)\, e^{2\pi i <a_j,t>} \quad (t,h) \in \bar{W} \times S_j, \quad 1 \leq j \leq m.$$

Proof of Theorem 3.1. It is sufficient to prove that if 0 is an interior point in E and $f(0) = 1$, then f is equal to a local character in a nbd of 0. By a well known theorem in structure theory (see e.g. [8]. Theorem 24.30) the group G is (isomorphic with) the direct product of $\mathbb{R}^m$ and a LCA group G_0 containing a compact open subgroup. By Theorem 2.2. there is then a morphism $\psi: \mathbb{R}^q \to G_0$ and open connected nbds U and V of 0 in $\mathbb{R}^m$ and $\mathbb{R}^q$ respectively such that the nbd $U \times (\psi(V) + H)$ is contained in the interior of E and the morphism

$$\sigma: (x,t,h) \to (x, \psi(t) + h), \quad (x,t,h) \in \mathbb{R}^m \times \mathbb{R}^p \times H$$

restricted to $U \times V \times H$ is a local isomorphism. By Lemma 3.1 and its remark, there is an open nbd S of 0 in H such that $f \circ \sigma$ restricted to $U \times V \times S$

coinsides with a character of $\mathbb{R}^m \times \mathbb{R}^q \times H$. This proves the theorem. ∎

4. A stronger local result.

We will in this section show that in a sense to be made precise below, the number of local characters occuring in Theorem 3.1 is finite. The result will be given in terms of the concept of germs of local characters that we now introduce. If Ω is the set of all local characters in G the relation "there is a nbd of 0 in G in which $\theta \in \Omega$ and $\theta' \in \Omega$ are defined and agree" is an equivalence relation. The germs of local characters will be the equivalence classes of this relation.

Theorem 4.1. Let $f \in \mathcal{B}(E)$. Then f agrees locally in the interior of E with translates of local characters belonging to at most p different germs times complex numbers of modulus one. Here $p \leq C = \sup_{n \in \mathbb{Z}_+} \|f^n\|_{B(E)}$.

Proof. Let $x_1, \ldots, x_p$ be any p points in the interior of E. By Theorem 3.1 there are a nbd W of 0 in G, local characters $\theta_1, \ldots, \theta_p$ and complex numbers $c_1, \ldots, c_p$ of modulus one such that

$$(1) \qquad f(x+x_k) = c_k\, \theta_k(x), \quad x \in W \qquad 1 \leq k \leq p.$$

If we can show that

$$(2) \qquad \theta_1, \ldots, \theta_p \text{ all belongs to different germs}$$

implies that $p \leq C$, the theorem will follow. Assume (2) and regard G as $\mathbb{R}^m \times G_0$ where G_0 contains a compact open subgroup. We may suppose that W is so small that for $1 \leq k \leq p$, $\theta_k(W) \subset \{e^{2\pi i \tau} \mid |\tau| \leq \frac{1}{3}\}$ and hence

$$(3) \qquad \theta_k = 1 \text{ on each subgroup of } G \text{ contained in } W.$$

By Theorem 2.2. there is a morphism $\psi\colon \mathbb{R}^q \to G_0$, cubes $Q_q =]-r,r[^q$ and $Q_m =]-r,r[^m$ and a compact subgroup H such that the set $Q_m \times (\psi(Q_q) + H)$ is open and contained in W. Consider the morphism

$$\sigma\colon (y,t,l) \to (y, \psi(t)) + \sum_{k=1}^{p} l_k x_k \qquad y \in \mathbb{R}^m,\ t \in \mathbb{R}^q,\ l = (l_1, \ldots, l_p) \in \mathbb{Z}^p.$$

If e_k is the element in $\mathbb{Z}^p$ with its k:th coordinate equal to 1 and the

others equal to 0 and if $S = \bigcup_{k=1}^{p} (Q_m \times Q_q \times \{e_k\})$, then $\sigma(\overline{S}) \subset E$. Hence, as $\overline{S}$ is compact and σ is a morphism, it follows from Lemma 2.1 that the function $g = f \circ \sigma|_{\overline{S}}$ satisfies

(4) $\quad g \in A(\overline{S})$, $|g| = 1$ and $\|g^n\|_{A(\overline{S})} \leq C$, $n \in \mathbb{Z}_+$

Since the functions $(y,t) \to \theta_k(y, \psi(t))$, $(y,t) \in Q_m \times Q_q = Q$ are local characters in $\mathbb{R}^{m+q}$ and thus restrictions to Q of characters (cf Remark 2 in section 2) there are $d_k = (a_k, b_k) \in \mathbb{R}^{m+q}$ such that

$$\theta_k(y, \psi(t)) = e^{2\pi i(\langle y, a_k \rangle + \langle t, b_k \rangle)} \qquad (y,t) \in Q.$$

Hence, by (1)

(5) $\quad g(v, e_k) = c_k$ [illegible] $\quad v \in$ [illegible] $\quad 1 \leq k \leq p$.

By (2) and (3) all d_k are different and we will now use a slight modification of an argument due to Y. Domar (see [6], p. 4) to show that this together with (4) and (5) implies $p \leq C$.

Let $C' > C$ be given and choose $\phi_n \in L^1(\hat{\mathbb{R}}^s \times \mathbb{T}^p)$, $(s=m+q)$, such that $\|\phi_n\|_{L^1} \leq C'$ and $\hat{\phi}_n|_S = g^n$, i.e.

$$\hat{\phi}_n(v, e_k) = c_k^n \, e^{2\pi i n \langle d_k, v \rangle}, \quad v \in Q \qquad 1 \leq k \leq p.$$

If $\Delta(\xi) = \Delta_1(\xi_1) \cdot \ldots \cdot \Delta_1(\xi_s)$, $\xi = (\xi_1, \ldots, \xi_s) \in \hat{\mathbb{R}}^s$, where $\Delta_1(t) = \max(1 - |t|, 0)$, $t \in \mathbb{R}$, then $\check{\Delta} \in L^1(\mathbb{R}^s)$ and $\hat{\check{\Delta}} = \check{\hat{\Delta}} = \Delta$, where $\check{\Delta}(\xi) = \Delta(-\xi)$. For $d \in \mathbb{R}^s$, $\varepsilon > 0$ and $n \in \mathbb{Z}_+$ given we form the function

$$F_{d,\varepsilon,n}(v, l) = \int\!\!\int_{\mathbb{R}^s \times \mathbb{T}^p} \Delta\left(\frac{1}{\varepsilon}\left(\frac{\xi}{n} - d\right)\right) \phi_n(\xi, \eta) \, e^{-2\pi i \langle l, \eta \rangle} \, d\xi \, d\eta$$

on $\mathbb{R}^s \times \mathbb{Z}^p$, i.e. the Fourier transform of the function $(\xi, \eta) \to \Delta\left(\frac{1}{\varepsilon}\left(\frac{\xi}{n} - d\right)\right) \phi_n(\xi, \eta)$, $(\xi, \eta) \in \mathbb{R}^s \times \mathbb{T}^p$.

By a routin calculation, $(\xi, \eta) \to \Delta\left(\frac{1}{\varepsilon}\left(\frac{\xi}{n} - d\right)\right)$, $(\xi, \eta) \in \mathbb{R}^s \times \mathbb{T}^p$ is the Fourier transform of the function $h_{\varepsilon n} \in L^1(\mathbb{R}^s \times \mathbb{Z}^p)$ given by

$$h_{\varepsilon n}(u, m) = \begin{cases} [\text{illegible}] \, e^{2\pi i \langle [\text{illegible}], d \rangle}, \varepsilon n u) \Delta(\varepsilon n u) & \text{if } m = 0 \\ [\text{illegible}] & \text{if } m \neq 0 \end{cases}$$

Thus by Parseval's formula

$$F_{d,\varepsilon,n}(v, l) = \int_{\mathbb{R}^s} \sum_{m \in \mathbb{Z}^p} h_{\varepsilon n}(u, m) \, \phi_n(u+v, m+l) \, du = \int_{\mathbb{R}^s} h_{\varepsilon n}(u, 0) \, \phi_n(u+v, l) \, du$$

and hence

$$F_{d,\varepsilon,n}(v,1) = \int_{\mathbb{R}^s} \phi_n(\frac{u}{\varepsilon n} + v, 1)\, e^{2\pi i \langle \frac{d}{\varepsilon}, u\rangle}\, \Delta(u)du.$$

For n large $\frac{u}{\varepsilon n} \in Q$ and then by (5)

$$\hat{\phi}_n(\frac{u}{\varepsilon n};\, e_k) \cdot \hat{\phi}_n(0,e_k)^{-1} = e^{2\pi i n \langle d_k, \frac{u}{\varepsilon n}\rangle} = e^{2\pi i \langle d_k, \frac{u}{\varepsilon}\rangle},$$

which shows that the functions $u \to \hat{\phi}_n(\frac{u}{\varepsilon n},\, e_k)\hat{\phi}_n\,(0,e_k)^{-1}$ tends to the function $u \to e^{2\pi i \langle d_k, \frac{u}{\varepsilon}\rangle}$ pointwise in $\mathbb{R}^s$. By the Lebesgue's dominated convergence theorem, for $1 \le k \le p$,

$$\lim_{n\to\infty} F_{d,\varepsilon,n}\,(0,e_k) \cdot \hat{\phi}_n(0,e_k)^{-1} = \int_{\mathbb{R}^s} e^{2\pi i \langle \frac{d_k}{\varepsilon}, u\rangle} \cdot e^{2\pi i \langle \frac{d}{\varepsilon}, u\rangle} \hat{\Delta}(u)du =$$

$$= \Delta(\frac{d_k + d}{\varepsilon}).$$

As $\Delta(0) = 1$ and $|\phi_n(0,e_k)| = 1$, given $\delta > 0$ we can choose n large enough to have

$$\forall \varepsilon > 0 \quad |F_{-d_k,\varepsilon,n}(0,e_k)| \ge 1 - \frac{\delta}{p}, \qquad 1 \le k \le p.$$

With n choosen in this way, by the definiiton of $F_{d,\varepsilon,n}$,

$$p - \delta \le \sum_{k=1}^{p} |F_{-d_k,\varepsilon,n}(0,e_k)| \le \sum_{k=1}^{p} \iint_{\mathbb{R}^s \times \mathbb{T}^n} \Delta(\frac{1}{\varepsilon}(\frac{\xi}{n} + d_k))\, |\phi_n(\xi,\eta)|\, d\xi d\eta.$$

But $\Delta(\frac{1}{\varepsilon}(\frac{\xi}{n} + d_k)) = 0$ if not all coordinates of $\frac{\xi}{n} + d_k$ has modulus at most ε. As all d_k are different, with n fixed and large, we can choose ε so small that the functions $\xi \to \Delta(\frac{1}{\varepsilon}(\frac{\xi}{n} + d_k))$ $\xi \in \mathbb{R}^s$, $1 \le k \le p$ have disjoint supports. This implies

$$\sum_{k=1}^{p} \iint_{\mathbb{R}^s \times \mathbb{T}^n} \Delta(\frac{1}{\varepsilon}(\frac{\xi}{n} + d_k))\, |\phi_n(\xi,\eta)|\, d\xi d\eta \le \|\phi_n\|_{1,1} \le C'.$$

Thus $p \le C' + \delta$ and as $\delta > 0$ and $C' > C$ was arbitrary, we have proved that $p \le C$. ∎

5. The characterization.

In this section we will assume that E is compact and the closure of its interior. Let $\mathcal{N}(E)$ be the set of all complex-valued functions on E with the following property: for each $x \in E$ there are a nbd U of 0 in G, a local character $\theta: U \to T$ and a $c \in T$ such that

$$\forall y \in E \cap (x+U) \quad f(y) = c\theta(y-x).$$

We will now prove our main result, i.e. that $\mathcal{B}(E) = \mathcal{J}(E)$, and describe the structure of the functions in $\mathcal{J}(E)$. We start with a simple lemma concerning Diofantine approximation.

Lemma 5.1. Let $0 < \varepsilon \leq 1$ and $\tau_1, \ldots, \tau_k$ be real numbers such that

$$0 < \tau_1 < \frac{\varepsilon}{2} \quad \text{and} \quad 0 < \tau_j < \frac{\varepsilon}{2}\tau_{j-1}, \quad 2 \leq j \leq k.$$

Then there are infinitely many positive integers n such that the distance from $n\tau_j$ to the nearest odd integer is less then ε for $1 \leq j \leq k$.

Proof. Let h be any positive odd integer and n_k be the first positive integer for which $n_k\tau_k > h - \varepsilon$. Then for each integer l with $0 \leq l \leq [\frac{2\varepsilon}{\tau_k}]-2$, we will have $|(n_k+l)\tau_k - h| < \varepsilon$. As the length of the interval $I_{k-1} = [n_k\tau_{k-1}, (n_k+[\frac{2\varepsilon}{\tau_k}]-2)\tau_{k-1}]$ is at least $(\frac{2\varepsilon}{\tau_k} - 3)\tau_{k-1} > 4 - 3\tau_{k-1} \geq \frac{13}{4}$ there is an odd integer, say h_{k-1}, such that $[h_{k-1} - \varepsilon, h_{k-1} + \varepsilon] \subset I_{k-1}$. If $n_{k-1} = n_k + l_{k-1}$ is the first integer among the integers $n_k + l$ $0 \leq l \leq [\frac{2\varepsilon}{\tau_k}] - 2$ for which $n_{k-1}\tau_{k-1} > h_{k-1} - \varepsilon$, we will have $|(n_{k-1}+l)\tau_{k-1} - h_{k-1}| < \varepsilon$ whenever $0 \leq l \leq$ $\leq [\frac{2\varepsilon}{\tau_{k-1}}] - 2$. Thus the argument can be repeated. Continuing in this way we finally find at least 3 consecutive integers n among the integers $n_k + l$, $0 \leq l \leq [\frac{2\varepsilon}{\tau_k}] - 2$ and odd integers $h_1, \ldots, h_{k-1}$ such that

$$|n\tau_j - h_j| < \varepsilon, \quad 1 \leq j \leq k-1$$

Since h was arbitrary, the lemma is proved. ∎

Lemma 5.2. If U is an open set in G with $K = \overline{U}$ compact, if $f \in \mathcal{B}(K)$ and $f|_U$ is locally constant, then $f(K)$ is finite.

Proof. Suppose that $f(K)$ is infinite. As $f|_U$ is locally constant, there are a sequence $(x_m)_{m=1}^{\infty}$ in U and a sequence $(V_m)_{m=1}^{\infty}$ of compact symmetric nbds of 0 in G such that

$$2V_{m+1} \subset V_m, \quad x_m + V_m \subset U, \quad f(x) = f(x_m) \text{ on } x_m + V_m$$

and such that the points $f(x_m) \in T$ are distinct. Let $H = \bigcap_{m=1}^{\infty} V_m$, which is a

closed subgroup of G. For each nbd W of 0 in G there is a $k \in \mathbb{Z}_+$ such that $V_k \subset W + H$; otherwise, the sequence $V_k \setminus (W+H)$ of compact sets has the finite intersection property and hence a non-void intersection, which is impossible. Thus if $\alpha: G \to G/H$ is the canonical map the sets $\alpha(V_k) = \{x + H \mid x \in V_k\}$, $k \in \mathbb{Z}_+$ form a basis at the origin in G/H. Since K is compact, $\alpha(K)$ is compact and therefore the sequence $(x_m + H)_{m=1}^{\infty}$ in G/H has a convergent subsequence. By choosing a subsequence of $(x_m)_{m=1}^{\infty}$ if necessary we may thus assume that $(x_m+H)_{m=1}^{\infty}$ is convergent and

(1) if $c_m = f(x_m)$ and $e^{\pi i \tau_j} = \frac{c_{2j}}{c_{2j+1}}$, then $0 < \tau_1 < \frac{1}{4}$ and $0 < \tau_j < \frac{\tau_{j-1}}{4}$, if $j \geq 2$.

From (1) it follows by Lemma 5.1. that there is an infinite sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that

(2) $$\left|\left(\frac{c_{2j}}{c_{2j+1}}\right)^{n_k} - 1\right| \geq \frac{1}{2} \quad \text{if} \quad 1 \leq j \leq k.$$

Since $f \in \mathcal{B}(K)$ there is a bounded sequence $(\mu_k)_{k=1}^{\infty}$ in $M(\Gamma)$ such that $\hat{\mu}_k(x) = c_m^{n_k}$, if $x \in x_m + V_m$. Now let μ be a weak*-clusterpoint of $(\mu_k)_{k=1}^{\infty}$. For any finite set of functions $g \in L^1(G)$, vanishing outside $x_m + V_m$, it follows by Parsevals formula that there is a clusterpoint d_m of $(c_m^{n_k})_{k=1}^{\infty}$ such that

$$\int_G g(x)\,[\hat{\mu}(x) - d_m]\,dx = 0.$$

As $\hat{\mu}$ is continuous, this implies that $\hat{\mu}(x) = d_m$ in the interior of $x_m + V_m$ and by (2) we have

$$\left|\frac{d_{2j}}{d_{2j+1}} - 1\right| \geq \frac{1}{2}, \quad j \in \mathbb{Z}_+.$$

But if V is the interior of the set $\bigcup_{m=2}^{\infty}(x_m + V_{m+1})$, the function $h: x + H \to \hat{\mu}(x)$, $x \in V$ is well defined and uniformly continuous on $\alpha(V)$. As $h(x_m+H) = d_m$ and $(x_m+H)_{m=1}^{\infty}$ converges as $m \to \infty$ we have reached a contradiction. ∎

Theorem 5.1. If E is compact and the closure of its interior then $\mathcal{B}(E) = \mathcal{A}(E)$.

Proof. Suppose that $f \in \mathcal{B}(E)$ and let U be the interior of E. By Theorem

4.1. there are p local characters $\theta_k : W_0 \to T$ such that the open sets

$$U_k = \{x \in U \mid \exists \text{ nbd } W \text{ of } 0 \text{ in } G \text{ s.t. } W \subset W_0,\ x + W \subset U \text{ and } \forall y \in x + W,\ f(y) = f(x)\,\theta_k(y-x)\}$$

form a partition of U.

Let $x_0 \in \overline{U}_k$ and let V_0 be an open nbd of 0 in G with $\overline{V}_0$ compact and contained in W_0. The function

$$x \to \frac{f(x)}{\theta_k(x-x_0)}, \qquad x \in \overline{(x_0+V_0) \cap U_k}$$

is in $\mathcal{B}(\overline{(x_0+V_0) \cap U_k})$ and is locally constant in $(x_0+V_0) \cap U_k$. Hence, by Lemma 5.2, it has a finite range and this implies that the function is constant in $\overline{U}_k \cap (x_0+V)$, for some nbd V of 0 in G. Thus $\mathcal{B}(E) \subset \mathcal{Z}(E)$. Since E is compact we have $\mathcal{Z}(E) \subset \mathcal{A}(E)$ (cf the introduction). ∎

Remark. The proof actually shows that if E is the closure of its interior (compact or not) and if $f \in \mathcal{A}(E)$ then f agrees locally on E with translates of $p \leq \sup_{n \in \mathbb{Z}_+} \|f^n\|_{B(E)}$ local characters belonging to different germs times constants of modulus one. However, a function with such a representation need not even belong to $B(E)$ if E is non-compact. As an example the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 1 \end{cases}$$

does not belong to $B(]-\infty,0] \cup [1,\infty[)$. If it did, there would be a measure $\mu \in M(\mathbb{R})$ with $\hat{\mu}(x) = 0$ if $x \leq 0$ and $\hat{\mu}(x) = 1$ if $x \geq 1$. But by the F. and M. Riesz theorem (see eg [12], Theorem 8.2.7), a $\mu \in M(\mathbb{R})$ with $\hat{\mu}(x) = 0$ if $x \leq 0$ is absolutely continuous. Hence by the Riemann-Lebesgues lemma $\lim_{x \to +\infty} \hat{\mu}(x) = 0$, a contradiction.

The rest of this section will be devoted to an explicit description of the functions in $\mathcal{Z}(E)$ for E which is compact and the closure of its interior. We start with a definition.

Definition 5.1. If U is an open nbd of 0 in a LCA group G, a function $\omega : U \to T$ is said to be a local quasi-character if there are open nbds $V \subset W \subset U$ of 0 in G such that

(1) $\omega|_W$ is a local character

(2) $\forall x \in U \quad \exists x_1, \ldots, x_m \in W$ s.t. $x = \sum_{i=1}^{m} x_i$ and $\omega(x) = \prod_{i=1}^{m} \omega(x_i)$

(3) $\forall x \in U \quad \forall x' \in V \quad x + x' \in U \Rightarrow \omega(x+x') = \omega(x)\,\omega(x')$.

The next simple lemma shows that the class of local quasi-characters coincides with that of functions in G, induced of characters of groups, locally isomorphic with G. More precisely

Lemma 5.3. The function $\omega: U \to T$ is a local quasi-character in G if and only if there are a LCA group G', an open morphism $\alpha: G' \to G$ a $\beta \in \hat{G}'$ and open nbds $V' \subset W' \subset U'$ of 0 in G' such that

(1') $\alpha|_{W'}: W' \to \alpha(W')$ is a local isomorphism of G' with G.

(2') U' is contained in the subgroup of G' generated by W'.

(3') α is injective on $U' + V'$, $\alpha(U') = U$ and $\omega = \beta o (\alpha|_{U'})^{-1}$.

Proof. Suppose that $\omega: U \to T$ is a local quasi-character. We introduce the group $\tilde{G}$ and the maps α and β constructed from G and the local character $\omega|_W$ in Theorem 2.1. If $\tilde{U}$ is the graph of ω, then $\tilde{U} \subset \tilde{G}$ by (2) in Definition 5.1. From (3) it follows that $\tilde{U}$ is open in $\tilde{G}$, that $\alpha(\tilde{U}) = U$ and that $\omega = \beta o (\alpha|_{\tilde{U}})^{-1}$. Furthermore (3) implies that if $\tilde{V}$ is the graph of $\omega|_V$, then $(\tilde{U} - \tilde{U} + \tilde{V}) \cap \mathrm{Ker}\alpha = \{0\}$. Hence α is injective on $\tilde{U} + \tilde{V}_1$, for each open nbd $\tilde{V}_1$ of 0 in $\tilde{G}$ that satisfies $\tilde{V}_1 - \tilde{V}_1 \subset \tilde{V}$. This proves the necessity. The sufficiency is established by a straightforward inspection. ∎

It is readily seen that a local quasi-character is a uniformly continuous function. If $\omega: U \to T$ is a local quasi-character and $\bar{U}$ is compact, then $\omega \in \mathcal{B}(\bar{U})$ by Theorem 2.3. and the preceeding lemma. However, if $\bar{U}$ is non-compact, ω will in general not even belong to $B(\bar{U})$. We will have example on this in the next section where a main point is to seek conditions, less restrictive than compactness of $\bar{U}$, which imply that $\omega \in \mathcal{B}(\bar{U})$.

Theorem 5.2. Suppose that E is a compact set which is the closure of its interior. Then $f \in \mathcal{L}(E)$ if and only if there are a partition of E into disjoint compact sets $E_1, \ldots, E_m$, each E_i contained in an open set U_i, $x_i \in E$ and local quasi-characters $\omega_i : U_i - x_i \to T$ and $c_i \in T$, $1 \leq i \leq m$, such that

$$f(x) = c_i \omega_i(x - x_i), \quad x \in E_i.$$

Proof. As it is immediately seen that a function with the indicated representation is in $\mathcal{L}(E)$ it is enough to show the converse statement. Suppose that $f \in \mathcal{L}(E)$. From the definition of $\hat{\mathcal{L}}(E)$ it follows, by compactness, that there are $x_1, \ldots, x_n \in E$, open nbds $V_1, \ldots, V_n$ of 0 in G, local characters $\theta_i : V_i \to T$ and $c_i \in T$ such that

$$(1) \qquad E \subset \bigcup_{i=1}^{n} (x_i + V_i)$$

and

$$(2) \qquad f(x) = c_i \theta_i(x - x_i) \text{ if } x \in E \cap (x_i + V_i).$$

Let Q be the interior of E and $Q_i = Q \cap (x_i + V_i)$. Assume that $\overline{Q}_i \cap \overline{Q}_j \neq \emptyset$ and choose a $x_0 \in \overline{Q}_i \cap \overline{Q}_j$. As $x_0 \in E$ we have $x_0 \in x_k + V_k$ for some k with $1 \leq k \leq n$ by (1) and since $x_k + V_k$ is a nbd of x_0, we have

$$(x_k + V_k) \cap Q \cap (x_i + V_i) \neq \emptyset, \qquad (x_k + V_k) \cap Q \cap (x_j + V_j) \neq \emptyset.$$

Since these intersections are open, (2) implies that θ_i and θ_j belong to the same germ. Hence if $g_1, \ldots, g_p$ are all the different germs to which the local characters θ_i $1 \leq i \leq n$ belong and $M_j = \{i \mid \theta_i \in g_j\}$ then E is decomposed into the disjoint compact sets $E_j = \bigcup_{i \in M_j} \overline{Q}_i$, $1 \leq j \leq p$. If $x \in E_j$ then $x \in \overline{Q}_i = \overline{[Q \cap (x_i + V_i)]}$ for some $i \in M_j$ and by (1), there is a k such that $x \in x_k + V_k$. As $x_k + V_k$ is a nbd of x we have $(x_k + V_k) \cap Q \cap (x_i + V_i) \neq \emptyset$, which implies that θ_i and θ_k are in the same germ. Thus $k \in M_j$ and therefore

$$(3) \qquad E_j \subset \bigcup_{i \in M_j} (x_i + V_i).$$

By using a refinement of the cover $(x_i + V_i)$ of E_j if necessary, we may

assume that there is an open nbd W of 0 in G and a local character $\varkappa_j: W_j \to$ such that $\bigcup_{i\in M_j} \overline{V}_i \subset W_j$ and $\theta_i = \varkappa_j$ on V_i, $i \in M_j$.

Now let G_j be the open subgroup of G generated by W_j. If $K_{j1}, \ldots, K_{jq}$ are all the different cosets of G_0 to which x_i, $i \in M_j$ belongs and $M_{jk} = \{i \in M_j \mid x_i \in K_{jk}\}$, then E_j is decomposed into the disjoint compact sets $E_{jk} = E_j \cap K_{jk}$, $1 \le k \le q$. Since $E_{jk} = \bigcup_{i\in M_{jk}} \overline{Q}_i \subset \bigcup_{i\in M_{jk}} (x_i + V_i)$, to finish the proof, it is sufficient to show: if $M \subset M_{jk}$ and f coincides with a local quasi-character on the compact nbd $F = \bigcup_{i\in M} \overline{Q}_i$ of 0 in G and F meets $\overline{Q}_l$, $l \in M_{jk}$ then f coincides with a local quasi-character on $F \cup \overline{Q}_l$.

Let $\widetilde{G}_j$ be the group and α_j and β_j be the maps constructed from the local character $_j: W_j \to T$, as in Theorem 2.1. If $\widetilde{F}$ and $\widetilde{N}$ are the graphs of $f|_F$ and $f|_{\overline{Q}_l}$ respectively, the hypothesis implies that $\widetilde{F}$ and $\widetilde{N}$ are compact sets in $\widetilde{G}$ and

$$f \circ \alpha_j(\tilde{x}) = \begin{cases} \beta_j(\tilde{x}) & \text{if } \tilde{x} \in \widetilde{F} \\ C_l \beta_j(\tilde{x}) & \text{if } \tilde{x} \in \widetilde{N} \end{cases}$$

for some constant $C_l \in T$. By the definition of α_j and β_j,

(3) $\quad \operatorname{Ker} \alpha_j \cap \operatorname{Ker} \beta_j = \{0\}$.

Since $\alpha_j(\widetilde{F}) \cap \alpha_j(\widetilde{N}) = F \cap \overline{Q}_l \neq \emptyset$, by (3) there is a unique $\tilde{h} \in \operatorname{Ker} \alpha$ such that $\tilde{h} + \widetilde{N}$ meets $\widetilde{F}$. As this $\tilde{h}$ satisfies $\beta_j(\tilde{h}) = C_l$, it follows that

(4) $\quad f \circ \alpha_j(\tilde{x}) = \beta_j(\tilde{x})$, if $\tilde{x} \in \widetilde{F} \cup (\tilde{h} + \widetilde{N})$.

By (3) it follows from (4) that α_j is injective on $\widetilde{F} \cup (\tilde{h} + \widetilde{N})$. Since this set is compact, there is a nbd $\widetilde{V}$ of 0 in $\widetilde{G}$ such that α_j is injective on $[\widetilde{F} \cup (\tilde{h} + \widetilde{N})] + \widetilde{V}$. If $\widetilde{V}_1$ is a nbd of 0 in $\widetilde{G}$ with $2\widetilde{V}_1 \subset \widetilde{V}$ and $\widetilde{U} = \widetilde{F} + (\tilde{h} + \widetilde{N}) + \widetilde{V}_1$ then Lemma 5.3. implies that $\beta_j \circ (\alpha_j|_{\widetilde{U}})^{-1}$ is a local quasi-character on $\alpha_j(\widetilde{U})$, which by (4) coincides with f on $F \cup \overline{Q}_l$. ∎

6. Some results for non-compact sets.

In all of this section it will be assumed that E is the closure of its interior and, in general, is not compact. We first show that some additional conditions on E is necessary to have a representation of the functions in $\mathcal{B}(E)$ resembling that in Theorem 5.2., with a finite numbers of constants. One such condition is that the interior of E is connected. In that case we show that a function in $\mathcal{B}(E)$ coincides on E with a translate of a local quasi-character, times a complex number of modulus one.

However, local quasi-characters are in general not even restrictions of Fourier-Stieltjes transforms. We end this section with a discussion of the contitions on a closed nbd E of 0 in G, any under which local quasi-character defined on a nbd of E belongs to $\mathcal{B}(E)$. We begin with some examples.

Example 6.1. Let G be a discrete group and E a Sidon set in G (c.f. [12], p. 121). Then there is a constant C such that $f \in L^\infty(E) \Rightarrow f \in B(E)$ and $\|f\|_{B(E)} \leq C\|f\|_{L^\infty(E)}$. Thus any $f \in L^\infty(E)$ with $|f| = 1$ on E is in $\mathcal{B}(E)$.

The next two examples depend on properties of certain discrete subsets in LCA groups, studied in detail in [9]. A subset L of a LCA group G is a coherent set of frequencies if there exists a $C > 0$ and a compact set K in $\Gamma = \hat{G}$ such that for each trigonometric polynomial $P(\gamma)$ with frequencies in L we have $\sup_{\gamma\in\Gamma} |P(\gamma)| \leq C \sup_{\gamma\in K} |P(\gamma)|$. The following result, due to Y. Meyer, is the reason for our interest for coherent sets of frequencies. We refer to [9], p. 126 for the proof.

Theorem 6.1. Let L be a coherent set of frequencies in G and assume that $\Gamma = \hat{G}$ is metrizable and separable. Then there is a compact nbd V of 0 in such that the sets $l + V$, $l \in L$ are pairwise disjoint and such that if H is the homeomorphism of $L + V$ onto $L \times V$ given by $H(l+t) = (l,t)$, $l \in L$, $t \in V$, there is $C > 0$ such that

$$g \in B(L \times V) \Rightarrow g \circ H \in B(L+V) \quad \text{and} \quad \|g \circ H\|_{B(L+V)} \leq C \|g\|_{B(L\times V)} .$$

Example 6.2. Let G be a LCA group containing a discrete subgroup H and assume that Γ is metrizable and separable. Let S be a Sidon set in H and

$f \in L^\infty(S)$. Since S is a coherent set of frequencies in G, by Theorem 6.1, there is a compact nbd V of 0 in G such that the function g defined on $E = S + V$ by

$$g(x) = f(1) \quad \text{if} \quad x \in 1 + V, \; 1 \in S$$

is in $\mathcal{B}(E)$.

A set S of real numbers is a topological Sidon set if each $f \in L^\infty(S)$ is the restriction to S of the Fourier-Stieltjes transform of a $\mu \in M(\mathbb{R})$ (see [9], p. 187). As an example, the set $L = \{2^n | n \in \mathbb{Z}_+\}$ is a topological Sidon set (see, [9], Theorem II). We finish our set of examples on functions in $\mathcal{B}(E)$ with those immediatly derived from the following result, borrowed from [9] (see Theorem VI, p. 192).

Theorem 6.2. Let L be a topological Sidon set of real numbers and d_L the infimum of the distances between two distinct point of L. For each d in $]0, \frac{d}{2}[$ there is a constant $C(d)$, which is an increasing function of d, such that any bounded function

$$b: L + [-d,d] \to \mathbb{C} \quad \text{which is constant on each interval } [1-d, 1+d] \text{ is}$$

the restriction to $L + [-d,d]$ of the Fourier transform of a measure whose norm does not exceed $C(d) \sup_{l \in L} |b(l)|$.

We will now assume that U, the interior of E, is connected. Without loss of generality we may also assume that U is a nbd of 0 in G, that $f(0) = 1$ and that the group G is connected. It is then of the form $\mathbb{R}^m \times K$ where $m \geq 0$ and K is a compact connected group (cf. eg [8], Theorem 9.8).

Theorem 6.3. Let U be an open connected nbd of 0 in the connected LCA group G and let $E = \overline{U}$. Suppose that $f \in A(E)$ and $f(0) = 1$. Then there are a connected LCA group G', an epimorphism $\alpha: G' \to G$, a $\beta \in G'$ and an open connected nbd U' of 0 in G' such that with $E' = \overline{U'}$

(1) there is a $g \in B(G')$ with $g = 1$ on E' and $g = 0$ on $E' + h$ for each $h \in \operatorname{Ker} \alpha$, $h \neq 0$.

(2) $\quad \operatorname{Ker}\alpha \cap \operatorname{Ker}\beta = \{0\}$

(3) $\quad E = \alpha(E')$ and $f = \beta \circ (\alpha|_{E'})^{-1}$.

Remark. Since g is uniformly continuous, (1) implies that there is a nbd W' of 0 in G' such that α is injective on $E' + W'$. Hence, by Lemma 5.3, the theorem shows in particular that if $f \in \mathcal{A}(E)$, then f coincides on E with a local quasi-character.

Proof. As U is connected there is, by Theorem 4.1., a local character $\theta: W_0 \to T$ such that for each $x \in U$, there is a nbd W_k of 0 in G with $W_x \subset W_0$, $x + X_x \subset U$ and such that $f(x+y) = f(x)\,\theta(y)$ for all $y \in W_x$. Let U_1 be the set of all $x \in U$ for which there are $x_1, \ldots, x_k \in W_0$ such that $x = \sum_{j=1}^{k} x_i$ and $f(x) = \prod_{i=1}^{k} \theta(x_i)$. (Of course, k may vary). Then $U_1 \supset W_0$ and if $x \in U_1$ and W_x is a nbd of 0 such that $W_x \subset W_0$, $x + W_x \subset U$ and $f(y) = f(x)\,\theta(y)$ for all $y \in W_x$, then trivially $x + W_x \subset V_1$. Hence U_1 is a non empty open subset of U. As it is equally simple to show that U_1 is closed in U, the connectedness of U implies that $U_1 = U$.

Let $\tilde{G}$ be the group and α and β be the maps constructed from the local character $\theta: W_0 \to T$ as in Theorem 2.1. and let $\tilde{U}$ be the graph of $f|_U$. By what have been showed above, $\tilde{U} \subset \tilde{G}$, $\alpha(\tilde{V}) = U$ and $f \circ \alpha = \beta$ on $\tilde{U}$. Since $\operatorname{Ker}\alpha \cap \operatorname{Ker}\beta = \{0\}$, which proves (2), this implies that $\alpha|_{\tilde{U}}$ is injective and thus $\alpha|_{\tilde{U}}: \tilde{U} \to U$ is a homeomorphism. Hence $\tilde{U}$ is connected and as $\tilde{G}$ is generated by $\tilde{U}$, it is also connected.

We will now prove (1) from which (3) easily follows. Since $f \in \mathcal{A}(E)$, the function $f \circ \alpha$ is in $\mathcal{A}(\alpha^{-1}(E))$ by Lemma 2.1. and as β is a character, $\frac{f\circ\alpha}{\beta} \in \mathcal{A}(\alpha^{-1}(E))$. Hence as $\bigcup_{\tilde{h} \in \operatorname{Ker}\alpha} (\tilde{U} + \tilde{h}) = \alpha^{-1}(U) \subset \alpha^{-1}(E)$, there is a $C > 0$ and a sequence $\mu_n \in M(\hat{\Gamma})$, where $\hat{\Gamma}$ is the dual group of $\tilde{\Gamma}$, such that,

$$\hat{\mu}_n(\tilde{x}) = \beta(\tilde{h})^{-n}, \text{ if } \tilde{x} \in \tilde{U} + \tilde{h}$$

and $\sup_{n \in \mathbb{Z}_+} \|\mu_n\|_{M(\hat{\Gamma})} \leq C$. If $\mu_0 = \delta_0$ and if

$$\nu_n = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \mu_k$$

then $\|\nu_n\|_{M(\widetilde{\Gamma})} \leq C$ and for each $\widetilde{x} \in \widetilde{U} + \widetilde{h}$

$$\hat{\nu}_n(\widetilde{x}) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \beta(\widetilde{h})^{-k} = \left(\frac{1+\beta(\widetilde{h})^{-1}}{2}\right)^n.$$

Since $\operatorname{Ker} \alpha \cap \operatorname{Ker} \beta = \{0\}$ it follows that

$$\lim_{n\to\infty} \hat{\nu}_n(\widetilde{x}) = \begin{cases} 1 & \text{if } \widetilde{x} \in \widetilde{U} \\ 0 & \text{if } \widetilde{x} \in \widetilde{U} + \widetilde{h},\ \widetilde{h} \in \operatorname{Ker} \alpha,\ \widetilde{h} \neq 0. \end{cases}$$

Let ν be a weak*-clusterpoint of $(\nu_n)_{n=1}^{\infty}$. As $\hat{\nu}$ is continuous and $\widetilde{U} + \widetilde{h}$, $\widetilde{h} \in \operatorname{Ker} \alpha$, is open it follows (cf. the proof of Lemma 5.2) that

$$\hat{\nu}(\widetilde{x}) = \begin{cases} 1 & \text{if } \widetilde{x} \in \overline{\widetilde{U}} \\ 0 & \text{if } \widetilde{x} \in \overline{\widetilde{U}} + \widetilde{h},\ \widetilde{h} \in \operatorname{Ker} \alpha,\ \widetilde{h} \neq 0. \end{cases}$$

■

Example 6.3. Let $G = \mathbb{R} \times T$ and $\alpha: \mathbb{R}^2 \to \mathbb{R} \times T$ be the map given by $\alpha(x,y) = (x, y+\mathbb{Z})$, $(x,y) \in \mathbb{R}^2$. Further let $g: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ bx & \text{if } x > 0,\ b \in \mathbb{R}_+, \end{cases}$$

and let $\beta \in \hat{\mathbb{R}^2}$ be defined by $\beta(x,y) = e^{2\pi i a y}$, $(x,y) \in \mathbb{R}^2$, where a is irrational. If

$$U' = \{(x,y) \in \mathbb{R}^2 \mid |y - g(x)| < \delta\}$$

and $\delta > 0$ is sufficiently small, then there is a nbd V' of 0 i $\mathbb{R}^2$ such that α is injective on $U' + V'$.

In this case, the function $f = \beta \circ (\alpha|_{U'})^{-1}$ is a local quasi-character, defined on $U = \alpha(U')$. However $f \notin B(\overline{U})$. If $f \in B(\overline{U})$, then $f \circ \alpha \in B(\alpha^{-1}(\overline{U}))$ by Lemma 2.1 and hence, by the theorem above, the function which is 1 on $\overline{U}'$ and 0 on $\overline{U}' + (0,k)$, $k \in \mathbb{Z}$, $k \neq 0$ would belong to $B(\overline{U}' + (\{0\} \times \mathbb{Z}))$. Since $\overline{U}' + (\{0\} \times \mathbb{Z})$ contains the set

$$\{(x,0) \in \mathbb{R}^2 \mid x \leq 0 \text{ or } x = \frac{k}{b},\ k \in \mathbb{Z}_+\}$$

this implies that the function which is 1 if $x \leq 0$ and is 0 if $x = \frac{k}{b}$, $k \in \mathbb{Z}_+$, would belong to $B(]-\infty, 0] \cup b^{-1}\mathbb{Z}_+)$. But this is impossible (cf. the remark following Theorem 5.1).

If the conditions (1), (2) and (3) of Theorem 6.3. are satisfied, then does the function $f = \beta o(\alpha|_{\overline{U}'})^{-1}$ belong to $B(\overline{\alpha(U')})$? We are not in position to give an elucidative necessary and sufficient condition on U' to have $f = \beta o(\alpha|_{\overline{U}'})^{-1}$ in $\mathcal{B}(\overline{\alpha(U')})$. As a corollary to the theorem below, we show that in a special case, the conditions (1), (2) and (3) are sufficient. For the theorem we need some new notations. If $\Gamma = \hat{G}$, we denote by $\mathcal{S}$ the subspace of $L^1(\Gamma)$ consisting of all functions $\phi \in (L^1 \cap C_0)(\Gamma)$ such that supp $\hat{\phi}$, the support of $\hat{\phi}$, is a compact subset of G. If V is an open subset in G, we denote by $\mathcal{S}(V)$ the set of all $\phi \in \mathcal{S}$ such that supp $\hat{\phi}$ is a compact subset of V. Further, if H is a closed subgroup of G, we denote its Haarmeasure by dh; similarly an Haarmeasure of the closed subgroup Λ of Γ is denoted by $d\lambda$.

Theorem 6.4. Let G be a LCA group with dual group Γ, let H be a discrete subgroup of G with annihilator Λ and $\alpha: G \to G/M$ be the canonical map. If U is an open nbd of 0 in G such that

(1) there is a $\nu \in M(\Gamma)$ such that $\hat{\nu}(x) = 1$, if $x \in \overline{U}$ and $\hat{\nu}(x) = 0$ if $x \in \overline{U} + h$, $h \in H$, $h \neq 0$.

(2) there is a sequence $(\psi_j)_{j=1}^{\infty}$ in $\mathcal{S}$ such that $\hat{\psi}_j(x) \to 1$ as $j \to \infty$ for each $x \in \overline{U}$ and $\sup_{\gamma \in \Gamma} \int_{\Lambda} |\psi_j(\gamma - \lambda)| \, d\lambda = C < \infty$,

then $f \in B(\overline{U}) \Rightarrow f o(\alpha|_{\overline{U}})^{-1} \in B(\overline{\alpha(U)})$ and $\|fo(\alpha|_{\overline{U}})^{-1}\|_{B(\overline{\alpha(U)})} \leq C_1 \|f\|_{B(\overline{U})}$

where $C_1 = C \cdot \|\nu\|_{M(\Gamma)}$.

Remark. Condition (2) could be weakened; however, in the application we have in mind, the stated result is sufficient.

The theorem follows easily from Poisson's formula and the lemmas below. For the first, we refer to [9], Proposition 8, p. 126; for the second see e.g. [7], Theorem 5.3.

Lemma 6.1. Let V be an open subset in G and f be a continuous function on $E = \overline{V}$. If there is a constant C such that

$$\forall \phi \in \mathcal{S}(V) \quad \left| \int_E f(x)\hat{\phi}(x)dx \right| \leq C \|\phi\|_{L^\infty(\Gamma)}$$

then $f \in B(E)$ and $\|f\|_{B(E)} \leq C$.

Lemma 6.2. Let H be a closed subgroup of G and $\alpha: G \to G/H$ the canonical map. Suppose that V is an open set in G/H, that $E = \overline{V}$ and that $f \in C(E)$. If $f \circ \alpha \in B(\alpha^{-1}(E))$, then $f \in B(E)$ and $\|f\|_{B(E)} \leq \|f \circ \alpha\|_{B(\alpha^{-1}(E))}$.

Proof of the theorem. Choose the Haarmeasure dh of H such that each point in H has mass 1. By (1) the set $\overline{U} + H$ is closed and the function g defined on $\overline{U} + H$ by

$$g(x) = \int_H f(x+h)\, \hat{\nu}(x+h)dh, \quad x \in \overline{U} + H$$

is well defined, continuous and equals f on $\overline{U}$. Hence by Lemma 6.2. it is sufficient to show that $g \in B(\overline{U}+H)$ and $\|g\|_{B(\overline{U}+h)} \leq C_1 \|f\|_{B(\overline{U})}$. For this, Lemma 6.1. will be used.

Let $\varepsilon > 0$ be given and choose a $\sigma \in M(\Gamma)$ such that

$$\hat{\sigma}(x) = f(x) \text{ if } x \in \overline{U}, \quad \|\sigma\|_{M(\Gamma)} \leq \|f\|_{B(\overline{U})} + \varepsilon.$$

Since $\psi_j \in \mathcal{S}$, we have $\psi_j * \sigma * \nu \in \mathcal{S}$. This implies that the restriction of $\psi_j * \sigma * \nu$ to Λ is in $L^1(\Lambda)$ (cf. e.g. [11], 5.2. (iii)). We form the measures $\mu_j \in M(\Gamma)$ defined by

$$\forall \phi \in C_0(\Gamma) \quad \langle \mu_j, \phi \rangle = \int_\Lambda \phi(\lambda)\, (\psi_j * \sigma * \nu)\, (\lambda)d\lambda.$$

If the Haarmeasure $d\lambda$ is normalized in such a way that Poisson's formula holds (see [11], 5.1.) we have (cf [11], 5.5. (ii) and 4.4)

$$\hat{\mu}_j(x) = \int_H (\psi_j * \sigma * \nu)\hat{}\,(x+h)dh = \int_H f(x+h)\, \hat{\nu}(x+h)\hat{\psi}_j(x+h)dh, \quad x \in U + H.$$

Thus $\hat{\mu}_j(x) \to g(x)$ as $j \to \infty$, for each $x \in U + H$, by (2). Moreover, by (2) we have

$$\|\mu_j\|_{M(\Gamma)} \leq \int_\Lambda |(\psi_j * \sigma * \nu)(\lambda)|\, d\lambda \leq \int_\Lambda \left[\int_\Gamma |\psi_j(\lambda - \gamma)|\, d|\sigma * \nu|\, (\gamma) \right] d\lambda \leq$$

$$\leq C \cdot \|\nu\| \cdot \|\sigma\|_{M(\Gamma)} \leq C \cdot \|\nu\| \cdot (\|f\|_{B(\overline{U})} + \varepsilon),$$

for each $j \in \mathbb{Z}_+$. Since $\varepsilon > 0$ was arbitrary we have $\|\mu_j\| \leq C \cdot \|\nu\|_{M(\Gamma)} \|f\|_{B(\overline{U})}$ for each $j \in \mathbb{Z}_+$. By using Parseval's formula, the theorem follows immediately from Lemma 6.1.

If $\theta: W_0 \to T$ is a local character in $G = \mathbb{R}^m \times K$, where K is compact and connected, there are nbds W_0' of 0 in $\mathbb{R}^m$ and W_0'' of 0 in K and local characters $\theta': W_0' \to T$ and $\theta'': W_0'' \to T$ such that $W_0' \times W_0'' \subset W$ and $\theta(x,y) = \theta'(x)\theta''(y)$ for all $(x,y) \in W_0' \times W_0''$. As $W_0' \subset \mathbb{R}^m$ we may assume that θ' is a restriction of a character. Thus the group $\tilde{G}$, used in the proof of Theorem 6.3, is (isomorphic with) $\mathbb{R}^m \times \tilde{K}$, where $\tilde{K}$ is the group constructed from K and the local character $\theta'': W_0'' \to T$.

If α is the usual epimorphism of $\tilde{G}$ onto G, its kernel H is of the form $H = \{0\} \times$ "H'' is a discrete supgroup of $\tilde{K}$. Thus the annihilator Λ of H in $\mathbb{R}^m \times (\tilde{K})^\wedge$ is of the form $\Lambda = \mathbb{R}^m \times \Lambda''$ where Λ'' is the annihilator of H'' in $(\tilde{K})^\wedge$. By the corollary below, it follows that if $\tilde{U} \subset \mathbb{R}^m \times \tilde{C}$ where $\tilde{C} \subset \tilde{K}$ is compact, the conditions (1), (2) and (3) are sufficient for $f = \beta o (\alpha|_{\tilde{U}})^{-1}$ to be in $\overline{\alpha(\tilde{U})} = \bar{U}$.

Corollary. Assume, with the notations of Theorem 6.4. that $G = \mathbb{R}^m \times G'$ and $H = \{0\} \times H'$. If there is a compact subset C' of G' such that $U \subset \mathbb{R}^m \times C'$ and if (1) of Theorem 6.4. is satisfied, then the conclusion of the theorem also holds.

Proof. The annihilator Λ of H is of the form $\mathbb{R}^m \times \Lambda'$ where Λ' is the annihilator of H'. Since $C' \subset G'$ is compact there is a $\phi \in \mathcal{D}(G')$ such that $\hat{\phi} = 1$ on C'. It is well known that there is a sequence $(\phi_j)_{j=1}^\infty$ in $\mathcal{D}(\mathbb{R}^m)$ such that $\hat{\phi}_j(x) \to 1$ as $j \to \infty$ for each $x \in \mathbb{R}^m$ and such that $\|\phi_j\|_{L^1(\hat{\mathbb{R}}^m)} = 1$ for each j. If ψ_j is defined on $\hat{\mathbb{R}}^m \times \Gamma'$, where $\Gamma' = \hat{G}'$, by $\psi_j(\xi,\gamma') = \phi_j(\xi)\,\phi(\gamma')$, then the sequence $(\psi_j)_{j=1}^\infty$ satisfies condition (2) of Theorem 6.4. ∎

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